B.Sc-part-I(H), Paper-I Abstract Algebra 18-07-2020

Group

Q.1. If a, b, c are three elements of a group (G, o) then

(I) $aoc = boc \Rightarrow a = b$ (Right cancellation laws)

II $coa = cob \Rightarrow a = b$ (left cancellation laws)

Proof: I $aoc = boc \Rightarrow (aoc)oc^{-1} = (boc)oc^{-1}$ [$\because c^{-1} \in G$]

$\Rightarrow ao(coc^{-1}) = bo(coc^{-1})$ (By associativity)

$\Rightarrow aoe = boe \Rightarrow a = b$

II $coa = cob \Rightarrow c^{-1}o(coa) = c^{-1}o(cob)$

$\Rightarrow (c^{-1}oc)oa = (c^{-1}oc)ob$

$\Rightarrow eoa = eob \Rightarrow a = b$

Q.2 In a group G, the equation $aox = b$ and $yoa = b$ where $a, b \in G$ have unique solution in G.

Proof: $aox = b$

$\Rightarrow a^{-1}o(aox) = a^{-1}ob$ [$\because a^{-1} \in G$]

$\Rightarrow (a^{-1}oa)ox = a^{-1}ob$ (By associativity)

$\Rightarrow eox = a^{-1}ob$

$\Rightarrow x = a^{-1}ob \in G$ [$\because a^{-1}, b \in G \Rightarrow a^{-1}ob \in G$]

Therefore, the equation $aox = b$ has a solution $x = a^{-1}ob$ in G.

Similarly, it can be proved that the equation $yoa = b$ has a solution $y = boa^{-1}$ in G.

Uniqueness:- Let x_1 and x_2 be any two solutions of the equation $aox = b$, so that

$aox_1 = b$ and $aox_2 = b$

$\Rightarrow aox_1 = aox_2$

$\Rightarrow x_1 = x_2$ (By left cancellation laws)

Therefore, the solution of the equation $aox = b$ its unique.

Similarly, it can be proved that the equation $yoa = b$

Hence, the given equations have unique solutions in G.

Proved

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Q.3. The left identity is also the right identity.

Proof: Let e be the left identity of a group G and let $a \in G$ be any element. Then

$$ea = a \quad \text{--- (1)}$$

To prove that e is also the right identity, it is sufficient to show that

$$ae = a \quad \text{--- (2)}$$

Let a^{-1} be the inverse of a, then

$$a^{-1}a = e \quad \text{--- (3)}$$

Now, $a^{-1}(ae) = (a^{-1}a)e = e \cdot e = e = a^{-1}a$

$\Rightarrow \quad a^{-1}(ae) = a^{-1}a$ ~~(By left cancellation law)~~

$\Rightarrow \quad ae = a$ (By left cancellation law)

Hence, we have that
The left identity is also the right identity of G.

Proved

Q.4 The left inverse of an element is also its right inverse.

Proof: Let a^{-1} be the left inverse of an element a of a group G, so that

$$a^{-1}a = e \quad \text{--- (1)}$$

where e is the identity of G

We have to prove that a^{-1} is also the right inverse of a, it is sufficient to show that

$$aa^{-1} = e \quad \text{--- (2)}$$

By associativity, we have

$$a^{-1}(aa^{-1}) = (a^{-1}a)a^{-1} = ea^{-1} \quad [\text{by (1)}]$$
$$= a^{-1} = a^{-1}e$$

$\Rightarrow \quad a^{-1}(aa^{-1}) = a^{-1}e$

$\Rightarrow \quad aa^{-1} = e$ [By left cancellation law]

Hence, from (2), we can say that the left inverse of an element is also the right inverse of that element.

Proved.

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